



SM358

Tutor-Marked Assignment 03

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Please send all your answers to the tutor-marked assignment (TMA) to reach your tutor by 12 noon (UK local time) on or before the cut-off date shown on the SM358 website. Your TMAs should be submitted through the eTMA system unless there are difficulties which prevent you from doing so. In these circumstances, you must negotiate with your tutor to get their agreement to submit your assignment on paper. The eTMA system allows for eTMA submission directly to the university 24 hours a day, and either gives you confirmation that your eTMA has been submitted successfully or, if there has been a problem, an error message informing you of the problem and what steps you can take to overcome it. If you submit online you must keep your receipt code in case you need to prove successful submission.

General information about policy and procedure is in the *Assessment Handbook* which you can access from your StudentHome page. However, there are a number of ways in which SM358 eTMA submission differs from what is described there. These are described in the document *Producing eTMAs for Level 3 physics and astronomy modules* on the SM358 website. See also the SM358 *Introduction and Guide* for module-specific information.

Of particular importance is the test submission, TMA 00. This will enable you to familiarize yourself with the system and allow your tutor to check that the format in which you save your TMAs is compatible with their own computer software. It is your responsibility to make sure that you submit documents in a compatible format and we strongly recommend that you submit TMA 00. TMAs submitted in an incorrect format may not be marked.

If you are submitting a paper copy, please allow sufficient time in the post for the assignment to reach its destination on or before the cut-off date. We strongly advise you to use first-class post and to ask for proof of postage. Do not use recorded delivery or registered post as your tutor may not be in to receive it. Keep a copy of the assignment in case it goes astray in the post. You should also include an appropriately completed assignment form (PT3). You will find instructions on how to fill in the PT3 form in the *Assessment Handbook*. Remember to fill in the correct assignment number (03).

This booklet provides advice about submission of TMA answers as well as the questions for TMA03. Although the marks for your assignments do not count directly towards your SM358 result, they are an essential part of your learning and you are required to engage satisfactorily with them. Please refer to the SM358 *Introduction and Guide*, for additional information about the module assessment.

Assignment cut-off dates

The cut-off dates for the assignments provide an important element of pacing for your study of SM358 and they are spread fairly uniformly through the year, leading up to the exam.

You should regard these dates as fixed points. *Any extension to a TMA cut-off date requires prior permission from your tutor, which may not always be given. Extensions may be granted in exceptional circumstances but it will never be possible to have an extension of more than 3 weeks.* Your tutor will, of course, be willing to discuss with you the best strategies for catching up if you have fallen behind, and should be able to help with questions if you are stuck.

Plagiarism

You are encouraged to discuss the SM358 materials and assignment questions with other students, but the answers to the assignment questions must be your own work. This does not stop you making judicious use of material from other sources, but you must acknowledge such use by giving the author's name, the year of publication, the name of the publication in which it appears (or the website address), and the edition or volume number and the page number. However, there is no need to give references for standard equations in the SM358 texts. You should read the University's guidelines on plagiarism in the *Assessment Handbook*, available online from your StudentHome page.

To check that all students are working in a fair and academically appropriate manner, the Open University is currently using some text-comparison software to detect potential cases of plagiarism in work that is submitted for assessment. Details of how this is implemented in this module are given on the SM358 website.

Further general advice on answering SM358 assignment questions is given in the first assignment booklet.

This assignment is related to Chapters 3–7 of Book 2. Your answers for this assignment will provide evidence of your achievement of many of the learning outcomes, as listed in the *Introduction and Guide*, Section 6.5. In particular, this assignment tests *Knowledge and understanding* outcomes 1–5, *Cognitive skills* outcomes 1–3 and 5 and all three of the *Key skills* outcomes.

Question 1

This question carries 35% of the marks for this assignment and relates mainly to Chapter 3 of Book 2, and particularly Achievements 1.6, 3.8, 3.10.

A spin- $\frac{1}{2}$ particle with spin gyromagnetic ratio $\gamma_s < 0$ is in a constant uniform magnetic field of magnitude B , pointing in the z -direction. The Hamiltonian matrix for this particle is

$$\hat{H} = \omega \hat{S}_z,$$

where $\omega = -\gamma_s B$ is the Larmor frequency.

(a) Suppose that at $t = 0$ the initial spin state of the particle is given by the spinor

$$|A\rangle_{\text{initial}} = \frac{1}{5} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Write down a spinor $|A\rangle$ that represents the spin state of the particle at any time $t > 0$. Use your answer to find expressions for the expectation values of S_x and S_y at any time $t > 0$. You may use the spin matrices

$$\hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{S}_y = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \quad (19 \text{ marks})$$

(b) Use the generalized Ehrenfest theorem to find expressions for $d\langle S_x \rangle/dt$ and $d\langle S_y \rangle/dt$ in terms of $\langle S_x \rangle$, $\langle S_y \rangle$, and ω . Show that the expectation values found in part (a) are consistent with these expressions. (16 marks)

Question 2

This question carries 30% of the marks for this assignment and relates to Chapter 4 of Book 2, and particularly to Achievements 4.6, 4.7, 4.9 and 4.10.

A pair of non-interacting particles is in a one-dimensional infinite square well with walls at $x = 0$ and $x = L$. Throughout the question, one particle is in the ground state with normalized energy eigenfunction

$$\psi_A(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) & \text{for } 0 \leq x \leq L \\ 0 & \text{elsewhere,} \end{cases}$$

and the other particle is in the first excited state with normalized energy eigenfunction

$$\psi_B(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) & \text{for } 0 \leq x \leq L \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Suppose that the particles are identical spin- $\frac{1}{2}$ fermions and that the two-particle system has total spin quantum number $S = 0$. Explain what this implies about the symmetries of the spin and spatial parts of the total wave function and hence write down an explicit expression for the spatial wave function $\psi(x_1, x_2)$ of the two-particle system at time $t = 0$. Your expression should include an appropriate normalization factor, but no proof of normalization is required. (9 marks)

(b) For the case discussed in part (a), write down an expression for the probability density that one particle (artificially labelled 1) is at x_1 and the other particle (labelled 2) is at x_2 . Hence calculate the probability that both particles are found in the left-hand side of the box (i.e. with position coordinates $0 \leq x_1 < L/2$ and $0 \leq x_2 < L/2$). You may use the integrals

$$\begin{aligned}\int_0^{L/2} \sin^2\left(\frac{\pi x}{L}\right) dx &= \frac{L}{4} \\ \int_0^{L/2} \sin^2\left(\frac{2\pi x}{L}\right) dx &= \frac{L}{4} \\ \int_0^{L/2} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{2\pi x}{L}\right) dx &= \frac{2L}{3\pi}.\end{aligned}$$

(10 marks)

(c) Suppose that the two identical fermions occupy the same two energy eigenstates as before, but that they are now in a two-particle state with total spin quantum number $S = 1$. Calculate the probability that both particles are found in the left-hand side of the box in this case. You need not include all steps of your working, but should indicate clearly how and why some signs in the working given for part (b) are modified in this case. (7 marks)

(d) Finally, suppose that the particles occupy the same two energy eigenstates, but that they are now identical spinless bosons. Would you expect the spatial distribution of this pair of particles to be similar to the identical fermions in part (b) or to the identical fermions in part (c)? Outline your reasoning. (4 marks)

Question 3

This question carries 35% of the marks for this assignment and relates to Chapters 6 and 7 of Book 2, and particularly to Achievements 6.8, 6.9 and 7.4.

Two photons moving in opposite directions along the y -axis are in the entangled polarization state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|VV\rangle + |HH\rangle),$$

where V denotes vertical polarization relative to the z -axis, and H denotes horizontal polarization relative to the z -axis.

The linear polarizations of the two photons are measured relative to different axes in the xz -plane. For photon 1, we use the axis \mathbf{n}_1 , whose direction is defined by $\theta = \theta_1$, where θ is the polar angle of spherical coordinates. For photon 2, we use the axis \mathbf{n}_2 , whose direction is defined by $\theta = \theta_2$. We define the following probabilities:

- P_{VV} is the probability that photon 1 gives vertical polarization relative to \mathbf{n}_1 and photon 2 gives vertical polarization relative to \mathbf{n}_2 .
- P_{HH} is the probability that photon 1 gives horizontal polarization relative to \mathbf{n}_1 and photon 2 gives horizontal polarization relative to \mathbf{n}_2 .
- P_{VH} is the probability that photon 1 gives vertical polarization relative to \mathbf{n}_1 and photon 2 gives horizontal polarization relative to \mathbf{n}_2 .
- P_{HV} is the probability that photon 1 gives horizontal polarization relative to \mathbf{n}_1 and photon 2 gives vertical polarization relative to \mathbf{n}_2 .

(a) Explain carefully why

$$P_{VV} + P_{HH} + P_{VH} + P_{HV} = 1.$$

(2 marks)

(b) Find an expression for P_{VV} . You may use the fact that

$$|V_\theta\rangle = \cos\theta|V\rangle + \sin\theta|H\rangle$$

for a state with vertical polarization relative to an axis in the xz -plane defined by the polar angle θ of spherical coordinates. Use trigonometric identities listed inside the back cover of the main text to express your answer as a function of the variable $\theta_2 - \theta_1$. (6 marks)

(c) For the polarization state $|\Psi\rangle$ considered in this question, it turns out that

$$P_{HH} = P_{VV} \quad \text{and} \quad P_{VH} = P_{HV}.$$

Combine these results with those derived in parts (a) and (b) to give explicit expressions for P_{HH} , P_{VH} and P_{HV} . Hence obtain a quantum-mechanical prediction for the correlation function

$$C(\theta_1 - \theta_2) = P_{VV} + P_{HH} - P_{VH} - P_{HV},$$

which represents the probability of agreement minus the probability of disagreement for the polarization measurements on the two photons. Express your answer as a simple function of the variable $2(\theta_1 - \theta_2)$. (5 marks)

(d) Explain briefly (in around 100 words) what is meant by a local hidden-variable theory. Suppose that the observer of the polarization of photon 1 chooses randomly between axes in the xz -plane defined by the polar angles $\theta_1 = 0^\circ$ and $\theta'_1 = 45^\circ$ of spherical coordinates, while the observer of the polarization of photon 2 chooses randomly between axes in the xz -plane defined by the polar angles $\theta_2 = 22.5^\circ$ and $\theta'_2 = -22.5^\circ$. Use an appropriate inequality to show that the quantum-mechanical prediction of part (c) is inconsistent with any local hidden-variable theory, explaining your reasoning carefully. (You need not prove or justify the inequality that you use.) (12 marks)

(e) Outline briefly (in around 300 words, plus suitable diagrams and equations) how a series of photon pairs prepared in the polarization state $|\Psi\rangle$ can be used to share a secure cryptographic key between two participants, Alice and Bob, using a protocol that makes essential use of entanglement. Your answer should describe which mutual choices of axes can be used to establish the key and explain how the presence of an eavesdropper could be detected. (10 marks)